Straightening almost chains in P(ω)

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Winter School in Abstract Analysis,
February 2, 2024

Separable compact lines

Definition

Consider a closed subset $F \subseteq [0,1]$, any set $X \subseteq F$ and define a space

$$F_X = F \times \{0\} \cup X \times \{1\}$$

equipped with the topology generated by the lexicographic order.

Theorem (Ostaszewski, 1974)

The space L is a separable compact linearly ordered space if and only if L is homeomorphic to F_X for some closed set $F \subseteq [0,1]$ and a subset $X \subseteq F$.

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Countable discrete extensions

Definition

Given a compact space K, we say that L is a countable discrete extension of K if the following are satisfied

1. K is a subspace of L,
2. L is compact,
3. $L \setminus K$ is a countable infinite discrete space.

Extension operators

Definition

For two compact spaces $K \subseteq L$ by an extension operator we mean a bounded linear operator $E : C(K) \to C(L)$ such that $Ef|K = f$ and for every $f \in C(K)$.

$\eta(K, L)$

For a compact space K and $L \in CDE(K)$ we are interested in the minimal norm of an extension operator $E : C(K) \to C(L)$. This value is denoted by $\eta(K, L)$.

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Old results

Theorem (Marciszewski)

There is a separable compact line K of weight ω_1 and $L \in CDE(K)$ such that $\eta(K, L) = 3$.

Theorem (K., Plebanek)

If $\kappa \geq non(\mathcal{E})$, then there is a separable compact line K of weight κ and $L \in CDE(K)$ such that $\eta(K, L) = \infty$.

$\mathcal{E}$ is the σ-ideal generated by closed measure zero sets;
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Old results 2

Theorem (K., Plebanek)

For a separable compact line K and $L \in CDE(K)$ we have $\eta(K, L) = 2k + 1$ for some $k \in \omega$ or $\eta(K, L) = \infty$.

Problem

[2, Problem 7.1.]

Is it relatively consistent that $\eta(K, L) < \infty$ for every separable compact space K of weight ω_1 and its countable discrete extension L?

Here we will focus on the situation where K is a separable compact line.

Almost chains

Almost chain of subsets of ω indexed by the set $X \subseteq [0,1]$ is a family $\mathcal{A}$ such that

- $\mathcal{A} = \{A_x \subseteq \omega : x \in X\}$,
- $A_x \subseteq^* A_y$ for $x < y$.

By almost chains we will always mean almost chains of subsets of ω (or other countable set) indexed by a set $X \subseteq [0,1]$ (which is usually fixed within a context).

Finite adjustment

We say that an almost chain $\mathcal{B}$ is a finite adjustment of $\mathcal{A}$ if for all $x \in X$ we have $A_x =^* B_x$.

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Alternations in an almost chain

Definition

We say that $\mathcal{A}$ is *barely alternating* if we cannot find $x_1 < x_2 < x_3 < x_4$ in X and $n \in \omega$ satisfying

$$n \in A_{x_1}, n \notin A_{x_2}, n \in A_{x_3}, n \notin A_{x_4}.$$

This property means that the almost chain $\mathcal{A}$ is alternating at most once in each n.

When for all n we cannot find any alternations, so there are no points $x < y$ in X such that $n \in A_x, n \notin A_y$, then the almost chain $\mathcal{A}$ is just a chain (with the regular inclusion).

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Earlier results

Theorem (Marciszewski, restated)

There is a set $X \subseteq [0,1]$ of cardinality ω_1 and an almost chain $\mathcal{A}$ which cannot be finitely adjusted into a non-alternating chain.

Problem for separable compact lines, restated

Is it relatively consistent that for every set $X \subseteq [0,1]$ of cardinality ω_1 and an almost chain $\mathcal{A}$ on X there is a finite adjustment $\mathcal{B}$ of $\mathcal{A}$ which has finite amount of alternations?

The straightening forcing

Theorem (Antonio Avilés)

(Under MA(κ)): Assume that we are given

- *a set $X \subseteq [0, 1]$ of cardinality κ,*
- *an almost chain $\mathcal{A} = \{A_x : x \in X\}$ of subsets of ω indexed by X.*

Then there is a barely alternating almost chain $\{B_x : x \in X\}$ which is a finite adjustment of $\mathcal{A}$, so for all $x \in X$ we have $A_x =^* B_x$.

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The scheme of proof

We use the following forcing:

$$\begin{aligned}\mathbb{P} = \{(F, \mathcal{B} = \{B_x \subseteq \omega : x \in F\}) :\ & F \subseteq X, F \text{ is finite}, \\ & A_x =^* B_x \text{ for } x \in F, \\ & \mathcal{B} \text{ is barely alternating}\}, \\ (F_1, \mathcal{B}_1) \leq (F_2, \mathcal{B}_2) \iff & F_1 \subseteq F_2 \wedge \mathcal{B}_1 \subseteq \mathcal{B}_2.\end{aligned}$$

It is then enough to prove the following:

1. $\mathbb{P}$ is ccc;
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Corollary

Under MA(κ), if K is a separable compact line of weight κ, then for each countable discrete extension L of K there is an extension operator $E : C(K) \to C(L)$ of norm at most 3.

References

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